

TRANSITION OF THE UNIFORM STATISTICAL FIELD ANYON STATE TO THE NONUNIFORM ONE AT LOW PARTICLE DENSITIES.

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Abstract

Using results of our exact description of the spinless fermion motion in a nonhomogeneous magnetic field $\mathbf{B} = B(0, 0, 1/\cosh^2(\frac{x-x_0}{\delta}))$ we study a gas of these particles moving in this field. For lower densities $\nu < \nu_c(B, \delta)$ the corresponding total energy is lower than that of the uniform field state. Thus when the density of anyons decreases a transition from the uniform statistical field state to the nonhomogeneous field state is predicted.

Non-Abelian anyons and topological quantum computation has recently emerged as one of the most exciting approaches to constructing a fault-tolerant quantum computer [1]. Strongly correlated quantum systems can exhibit behavior called topological order which is characterized by non-local correlations that depend on the system topology. Such systems can exhibit phenomena such as quasi-particles with anyonic statistics and have been proposed as candidates for naturally fault-tolerant quantum computation. Despite these remarkable properties, anyons have not been observed in the year 2008 directly [2]. Recently it was presented an experimental emulation of creating anyonic excitations in a superconducting circuit that consists of four qubits, achieved by dynamically generating the ground and excited states of the toric code model, i.e., four-qubit Greenberger-Horne-Zeilinger states. The anyonic braiding is implemented via single-qubit rotations: a phase shift of π related to braiding, the hallmark of Abelian $1/2$ anyons, has been observed through a Ramsey-type interference measurement [3].

Nevertheless it is interesting to study spinless fermion motion in a nonhomogeneous magnetic field $\mathbf{B} = B(0, 0, 1/\cosh^2(\frac{x-x_0}{\delta}))$ of these particles moving in this field. For lower densities $\nu < \nu_c(B, \delta)$ the corresponding total energy is lower than that of the uniform field state. Thus when the density of anyons decreases a transition from the uniform statistical field state to the nonhomogeneous field state is predicted. When the density of anyons present in this system decreases a transition from the uniform statistical field state to the nonhomogeneous field state is predicted. This may forward our observation possibilities to observe anyon properties.

An homogeneous magnetic field $\mathbf{B} = (0, 0, B)$ strongly influences states of charged fermions moving in an x-y plane perpendicular to the field direction. Landau energy levels and their degeneracy characterize this motion, [4]. A gas of free spinless fermions has its total energy $E_T(B, \nu)$ larger or equal to its total energy $E_T(0, \nu) = 2\pi t N \nu^2$ in the zero field:

$$\Delta E_h(n) \equiv E_T(B, \nu) - E_T(0, \nu) = 2\pi t N (\nu_{n+1} - \nu)(\nu - \nu_n) \quad (1)$$

where $\nu_{n+1} \geq \nu > \nu_n, \nu_n \equiv n \cdot \frac{\Phi}{\Phi_0}, n = 0, 1, \dots; \Phi_0$ is a unit of the magnetic flux, $\Phi \equiv B a^2, t \equiv \frac{\hbar^2}{2ma^2}$. Here ν is the number density of the gas, $\nu \equiv \frac{N_f}{N}$, N_f is the total number of fermions, $N a^2$ is the area of the square with the side length $L = a\sqrt{N}$, to which the motion is bounded, m is the fermion mass. Note that a is a characteristic length of the system, its value is of the order of a lattice constant value. The energy level degeneracy occurs only if a number of Landau levels is completely filled (e.i. if for some n $\nu_n = \nu$). Recently it was shown in [5] that in the presence of a periodic lattice potential the ground state energy of a gas of spinless fermions in an uniform magnetic field in the vicinity of the filled lowest Landau level is lower than that in zero field. This problem was studied

further in context of commensurate flux phases, [6]. If a nonhomogeneity of the field is introduced by a local field intensity decrease then competition of two tendencies is expected to occur: a decrease of the single fermion energy level due to decreased value of the field and a decrease of the every energy level degeneracy due to larger spacing between centers of neighboring orbits within the region of smaller fields. Spectrum of 2d Bloch electrons in a periodic magnetic field was studied in [7]. Using semiclassical methods authors of this later paper investigated the case where the magnetic unit cell is commensurate with the lattice unit cell. Their work is in some sense extension of previous studies of free electrons in periodic magnetic field [8] to the lattice case. Our aim in this paper is to present results of our study of the motion of a spinless fermion gas bounded to the square $L \times L$ in a nonhomogeneous static magnetic field perpendicular to this plane. We neglect the lattice periodic potential influence on the gas energy spectrum in this paper. We consider in more details the limit in which nonhomogeneity disappears and a uniform field appears. In difference to [7], [8] and [9] we do not consider a periodic magnetic field. Recently, [10], an exact description of motion of the quantum spinless fermion in a nonhomogeneous magnetic field described by the vector potential $\mathbf{A} = (0, B\delta \tanh(\frac{x-x_0}{\delta}), 0)$ was found. We use these results in this paper to study the stability of the statistical uniform anyon state with respect to a nonuniform field state. Firstly using the single fermion energy from [10] we find a total energy of a gas of spinless fermions moving in our nonhomogeneous field. Then we compare this energy with the total energy of the same gas moving in the uniform field with the same intensity B . We have found that at low densities $\nu < \nu_c(B, \delta)$ the nonhomogeneous field state of the anyon gas is preferred. Occurrence of such a kind of instability has consequences for interpretation of recent experiments [11] searching for the T- and P- symmetry breaking phenomena due to presence of particles with exotic statistics - anyons.

In the case of motion of a quantum spinless fermion in a nonhomogeneous magnetic field described by the vector potential $\mathbf{A} = (0, B\delta \tanh(\frac{x-x_0}{\delta}), 0)$ the energy spectrum of the motion in the x -direction is splitted, see in [10], into a discrete and a continuous parts for general values of the field B and of the nonhomogeneity parameter δ . We take $x_0 = 0$ in the following, thus field has its maximum intensity at $x = 0$. Let us consider the limit of strong fields ($F \equiv 2\pi \frac{\Phi'}{\Phi} \gg 1/2$, where $\Phi' \equiv B\delta^2$) in which case a linear type nonhomogeneity is localized near the two edges $x = \pm L/2$, if $\delta \gg a$ keeping L finite. In this limit it is sufficient to take into account the lowest energy levels of the spectrum. The eigenvalues of the energy corresponding to this part of the spectrum are given by, see in [10] :

$$E_n(p) = \frac{p_y^2}{2m} \left(1 - \frac{F^2}{((\frac{1}{4} + F^2)^{\frac{1}{2}} - ((1/2) + n))^2} \right) +$$

$$(\frac{\hbar^2}{2m\delta^2})(F^2 - ((\frac{1}{4} + F^2)^{\frac{1}{2}} - (\frac{1}{2} + n))^2,$$

where $n = 0, 1, \dots [n_{max}]$, here $[n]$ denotes an integer part of a real number n , p_y is the y-momentum. Let us define $P \equiv \frac{|p_y|\delta}{\hbar}$. The number n_{max} is defined by:

$$n_{max} = (\frac{1}{4} + F^2)^{\frac{1}{2}} - (1/2) - (|P| F)^{\frac{1}{2}},$$

for given values of P and F .

The limit of strong but still nonhomogeneous field is achieved for $F \rightarrow \infty$ keeping the nonhomogeneity parameter δ finite while increasing the field intensity B , $B \rightarrow \infty$. For $F^2 \gg \frac{1}{4}$ and for small quantum numbers n the energy $E_n(p_y)$ expanded into series of $1/F$ powers takes the form :

$$E_n(p_y) \approx \hbar\omega(n + \frac{1}{2}) - (\frac{\hbar^2}{2m\delta^2})((n + \frac{1}{2})^2 + \frac{1}{4}) - \frac{p_y^2}{mF}(n + \frac{1}{2}) + (\frac{\hbar^2}{8mF\delta^2})(n + \frac{1}{2}) + O(\frac{1}{F^3}).$$

where $\omega \equiv \frac{Bc}{em}$ is the cyclotron frequency. We see that the energy levels are degenerated in the limit of strong but modulated fields if the energy expansion above is restricted to the first two terms, which are of the F^1 and F^0 orders respectively. The largest value of the third term in this expansion is negligible with respect to the second term

$$max(\frac{p_y^2}{mF}(n + \frac{1}{2})) << (\frac{\hbar^2}{2mF\delta^2})(n + \frac{1}{2}).$$

if we take into account that there exists a natural cut-off for p_y momenta, $max(|p_y|) = \frac{\pi\hbar}{a}$, due to the underlying crystal and if we assume that the field intensity B satisfies the inequality:

$$\frac{\Phi}{\Phi_0} \gg 8\pi^2,$$

where $\Phi \equiv B.a^2$. If the third term and the following terms are not taken into account in calculations of the energy $E_n(p_y)$ then the degeneracy of the n -th level appears due to the lost of the energy dependence on p_y momentum. One can say that these levels are, [10], modified Landau levels with energies in the form:

$$E_n = \hbar\omega(n + \frac{1}{2}) - \frac{\hbar^2}{2m\delta^2}[(n + \frac{1}{2})^2 + \frac{1}{4}] + O(1/F), \quad (2)$$

where

$$n = 0, 1, \dots << n_m; n_m \approx F.$$

Note that

$$\frac{\hbar^2}{2m\delta^2} = 4t(\frac{L}{2\delta})^2/N.$$

From (2) we see that in the strong nonhomogeneous magnetic fields the neighboring energy levels are not equidistant as in the uniform field case. Every energy level E_n remains degenerated within considered approximation, its degeneracy D_n is found to be:

$$D_n = D_L \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}, \quad (3)$$

if the characteristic length L and the nonhomogeneity parameter δ satisfy

$$\tanh(L/2\delta) < (1 - \frac{2}{F}(n + \frac{1}{2})).$$

Here $D_L \equiv \frac{Bea^2}{\hbar c}N$ is the Landau level degeneracy as it is given in the case of the uniform field. The form of the degeneracy D_n given above holds for all orders of F . However, the large F expansion in (2) limits its validity to the region of system parameters given by the inequality below (3). This inequality follows from the usual, [12], boundary conditions: periodicity in the y -direction perpendicular to the x -axis and limits on the position of the orbit center in the x -direction to the region $-L/2, +L/2 > .$ The orbit center x -coordinate x_c is given, [10], by

$$\tanh(x_c/\delta) = (\frac{-p_y\delta}{\hbar})/F.$$

Note that this relation also reflects the fact that closed particle orbits of their motion in our magnetic field do exist only in the limited region of system parameters and of the p_y - *momentum* such that \tanh function above is note larger (or smaller) than 1 (than -1).

Straightforward calculations of the ground state energy $E_T(B, \delta, \nu)$ for spinless fermion gas with density ν in the limit of strong but nonhomogeneous fields specified by B, δ lead to the modification of (1). We have found that the energy difference between the nonhomogeneous field state and the zero field state:

$$\Delta E_{nh}(n) \equiv E_T(B, \delta, \nu) - E_T(0, \nu)$$

is given by the following expression:

$$\begin{aligned} \Delta E_{nh}(n) = & 2\pi tN[(\nu - \nu_n \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}})(\nu_{n+1} \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}} - \nu) + \\ & (1 - \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}})(\nu(2\nu_n + \nu_1) - \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}\nu_{n+1}\nu_n)] - \end{aligned} \quad (4)$$

$$-\frac{ta^2}{\delta^2}N[\nu(n^2 + n + \frac{1}{2}) - \nu_n(\frac{2n^2}{3} + n + \frac{1}{3})].$$

The total energy difference (4) is found assuming that there are n levels $0, 1, \dots, n-1$ filled and that the n -th level is filled partially. The gas density ν in (4) is limited by the following inequalities:

$$\nu_{n+1}\frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}} \geq \nu > \nu_n\frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}, \quad (5)$$

$$\nu_n \equiv n\Phi/\Phi_0.$$

The uniform field result (1) follows from (4) and (5) in the limit $\delta \rightarrow \infty$ keeping values of all the other system parameters constant.

When only the lowest energy level $n = 0$ is filled partially we find from (4) and (5) that:

$$\Delta E_{nh}(0) = 2\pi t N \nu (\nu_1 - \nu) - N \nu t (\frac{a}{\delta})^2 / 2, \quad (6)$$

where

$$\nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}} \geq \nu > 0.$$

The result (6) holds to the same order as the energy expansion (2). The filled lowest energy level $n = 0$ corresponds with the density ν given by:

$$\nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}} = \nu. \quad (7)$$

It follows from (7) that there is a decrease of the number of $n = 0$ states with respect to the uniform field case. In this later case the density at which the $n = 0$ state is filled is given by $\nu_1 \equiv \frac{\Phi}{\Phi_0}$. Moreover in our limit of large but still nonhomogeneous fields the quantity ν_1 satisfies the inequality given above (2).

Let us now compare total energies of our gas at a given density ν between the $n = 0$ state in the uniform field B and the $n = 0$ state in the nonhomogeneous magnetic field B with a finite parameter δ . We obtain from (1) and (6) that their total energy difference is given by:

$$E_T(B, \nu) - E_T(B, \delta, \nu) = N \nu t (\frac{a}{\delta})^2 / 2 > 0 \quad (8)$$

for

$$\nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}} \geq \nu > 0.$$

It follows from (8) that in this range of densities and of system parameters values the nonhomogeneous field state has lower energy than that in the homogeneous field.

Let us now increase the gas density ν to the value ν_1 . The lowest energy level of the uniform field state becomes filled. Let us assume that the nonhomogeneity parameter δ is large enough and such that the following inequalities hold:

$$\nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}} < \nu_1 < 2\nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}.$$

Then the $n = 0$ level of the nonhomogeneous field case is filled completely and the $n = 1$ level of the same case only partially. Let us compare energies of the uniform field state and of the nonhomogeneous state for the density $\nu_1 = \nu$. We obtain for the difference of the total energies of both states:

$$E^{n=1}(B, \delta, \nu) - E^{n=0}(B, \nu) = 4\pi t N \nu_1^2 \left(1 - \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}\right) - N \nu_1 t \left(\frac{a}{\delta}\right)^2 \left(5 - 4 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}\right) / 2.$$

This quantity is positive for macroscopically nonvanishing density ν . Decrease of the single fermion energy due to the nonhomogeneity is overcompensated by the decrease of the number of particles in the $n = 0$ nonhomogeneous field level and by their increase in the $n = 1$ level. The gap between energies of these two levels is

$$\hbar\omega - \frac{\hbar^2}{m\delta^2} + O(1/F),$$

the increase of the number of particles in the $n = 1$ level increases substantially the total energy of the system. Thus the uniform field state becomes preferred at higher particle densities. Qualitatively the same type of conclusions holds for higher densities ν and higher level numbers n .

We conclude that the nonhomogeneous field state of our gas of spinless fermions is preferred with respect to the uniform field state of the same gas for densities ν less or equal to a critical value $\nu_c(B, \delta)$ defined as

$$\nu_c(B, \delta) \equiv \nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}.$$

For higher densities $\nu > \nu_c(B, \delta)$ the later state is preferred.

One may ask at which value of the nonhomogeneity parameter δ the energy difference (8) takes the largest value. The difference $E_T(B, \nu) - E_T(B, \delta, \nu)$ from (8) becomes larger when $(\frac{L}{\delta})^2 = N(\frac{a}{\delta})^2$ is increasing quantity, e.i. when δ decreases with respect to the the length L . There exists a critical value of δ given by $\delta_c \equiv L/\ln(\pi N \nu_1)$. It is that limiting value of δ for which the inequality below (3) becomes equality. Below δ_c the degeneracy of every energy level becomes n -dependent [10] as it follows from p_y dependence of n_{max} given in the beggining of this paper. It is possible to find that

$$D_n \approx D_L(2\delta/L)(1 - \frac{2}{F}(n + \frac{1}{2})),$$

if

$$\tanh(L/2\delta) > 1 - \frac{2}{F}(n + \frac{1}{2}).$$

The density ν in (8) is for the $n = 0$ state now from the region:

$$\nu_1(2\delta/L)(1 - 1/F) \geq \nu > 0.$$

We have found that for such more localized nonhomogeneity of the field for which δ is smaller, $\delta < \delta_c$, the energy difference depends on $(L/2\delta)$ in the same way as in (8). The maximum value of this difference for the filled lowest level $n = 0$ of the nonhomogeneous field state is obtained for $\delta = a(3/2\pi\nu_1)^{1/2}$ as:

$$E_T(B, \nu) - E_T(B, \delta, \nu) = (4t\sqrt{N}/3)\nu\sqrt{\pi\nu_1}/6.$$

The gas density corresponding to this value is found to be $\nu = (\sqrt{6} - \frac{2}{3})\sqrt{\nu_1/\pi N}$. This density corresponds to nonzero linear density of particles, N_f/\sqrt{N} . Thus we conclude that in our type of the nonhomogeneous field the energy difference (8) is maximized for small densities of particles. One can say that it is an edge effect which leads to the maximum of the considered energy difference.

Let us discuss shortly consequences of our results obtained above for the anyon gas physics. According to [9] and [13] anyons may be described as spinless fermions moving in the statistical field generated by the statistical potential \mathbf{A}_i acting on the i -th anyon and given by:

$$\mathbf{A}_i = (1 - \rho)(\hbar c/e)\mathbf{z} \times \sum_{j \neq i} (\mathbf{r}_i - \mathbf{r}_j) / (|\mathbf{r}_i - \mathbf{r}_j|)^2.$$

The fractional statistics parameter is denoted here by ρ . A Hartree-Fock considerations in [9] lead to description of anyons with a single fermion Hamiltonian

$$H = \frac{1}{2m}[\mathbf{p} - \frac{e}{c}\mathbf{A}]^2,$$

where the uniform average statistical field is described by the vector potential $\mathbf{A} = (1 - \rho)(\hbar c\nu/a^2 e)\mathbf{z} \times \mathbf{r}$. This potential may be transformed into another gauge form:

$$\mathbf{A} = (1 - \rho)(\hbar c\nu/a^2 e)(0, x - x_0, 0).$$

We may assume that the potential of our field

$$\mathbf{A} = (0, B\delta \tanh((x - x_0)/\delta), 0)$$

is a result of the averaging of the potential \mathbf{A}_i given above over some stationary configurations of anyons which are different from those which lead to the uniform statistical field. Such a modulated form of the average potential may results from \mathbf{A}_i when fermions are non-homogeneously distributed within the plane.

From our results described above for the gas of spinless fermions moving in our nonhomogeneous field it follows that for the anyon densities ν such that

inequality

$$\nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}} \geq \nu > 0$$

holds (where ν, L, δ are fixed parameters) the uniform statistical field state is unstable with respect to the state with nonhomogeneous statistical field described by our potential. Our comparison of the total energies between the nonhomogeneous field state and the uniform field state as given above shows that the former is preferred at zero temperature. This instability effect of the uniform statistical field state is larger for smaller densities. However it is present also for macroscopically nonvanishing density of particles. For densities of anyons ν higher than $\nu_1 \frac{\tanh(\frac{L}{2\delta})}{\frac{L}{2\delta}}$ the uniform statistical field state is preferred for anyons. It is however not clear from our results whether our nonhomogeneous field is the most stable nonhomogeneous field state of anyons at lower densities of particles. In our calculations we may consider the characteristic length L to be either the linear dimension of the sample either it may be a characteristic length of a domain within the sample. In the later case the whole sample is expected to be covered by similar domains. From (8) it follows that the most preferred value of the nonhomogeneity parameter δ is that value for which $\nu = \nu_c(B, \delta)$ when the enrgy difference between both considered states becomes maximized.

Thus we see that our results are directly related to the physics of anyons. Experimental evidence for presence of these new physical phenomena in real materials is controversial nowadays. There exists some positive evidence for observation of broken T- and/or P- symmetry in superconductors based on oxidic layers, [11]. Some of these experimental results are interpreted, however, as a negative evidence or there is no their clear interpretation. Our results presented here may contribute to better understanding why there exists variety of different results obtained under different physical conditions and in different samples in cited above experiments. If the statistical field varies in the sample then also measured physical quantities such as the optical axis rotation angle will vary within the CuO_2 plane in cuprate perovskites as it may be found f.e. from the results of analysis in [14] of rotation of polarized light reflected from T- and P- violating phases. According to results presented in this paper in those samples in which higher densities of charge carriers occur the uniform statistical field anyon state may be realized. In those samples where the density of anyons is below some critical value ν_c the anyon statistical field becomes spatially modulated. Whether this modulation is described by the statistical field considered in our calculations remains an open problem. Here we have shown only that such a transition exists, from our results we are not able to say which type of field modulation represents the true ground state of anyons. We may expect that some other than our modulation may lead to energetically (we consider $T=0$) more preferred anyon state.

Results of this paper point to principal possibility that a phase transition between anyon states: uniform statistical field state and modulated statistical

field state occurs when the carrier density is decreased. Such a phase transition may be experimentally observed under appropriate conditions. Physical properties of modulated states as well as their response to external signals probing their nature should be established in order to improve our understanding of the experimental situation in search of broken T-/P- symmetries due to presence of anyons. It is known, [15], that dynamic response of fermions in continuum as well as on the lattice in a magnetic field may be calculated. This task is beyond the scope of this paper.

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